1.0 TANGENT LINES, VELOCITIES, GROWTH

In section 0.2, we estimated the slope of a line tangent to the graph of a function at a point. At the end of section 0.3, we constructed a new function which was the slope of the line tangent to the graph of a function at each point. In both cases, before we could calculate a slope, we had to **estimate** the tangent line from the graph of the function, a method which required an accurate graph and good estimating. In this section we will start to look at a more precise method of finding the slope of a tangent line which does not require a graph or any estimation by us. We will start with a nonapplied problem and then look at two applications of the same idea.

The Slope of a Line Tangent to a Function at a Point

Our goal is to find a way of exactly determining the slope of the line which is tangent to a function (to the graph of the function) at a point in a way which does not require us to have the graph of the function.

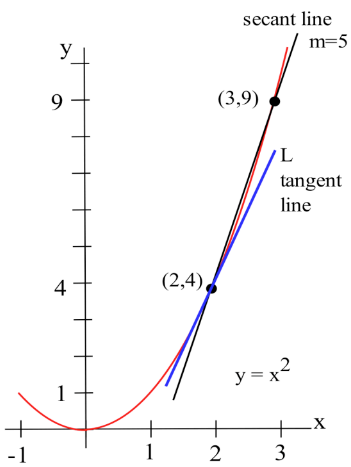

Fig. 1

Let's start with the problem of finding the slope of the line L (Fig. 1) which is tangent to $f(x) = x^2$ at the point (2,4). We could estimate the slope of L from the graph, but we won't. Instead, we can see that the line through (2,4) and (3,9) on the graph of f is an approximation of the slope of the tangent line, and we can calculate that slope exactly: $m = \Delta y/\Delta x = (9–4)/(3–2) = 5$. But $m = 5$ is only an estimate of the slope of the tangent line and not a very good estimate. It's too big. We can get a better estimate by picking a second point on the graph of f which is closer to (2,4) — the point (2,4) is fixed and it must be one of the points we use. From Fig. 2, we can see that the slope of the line through the points (2,4) and (2.5,6.25) is a better approximation of the slope of the tangent line at (2,4): $m = \Delta y/\Delta x = (6.25 – 4)/(2.5 – 2) = 2.25/.5 = 4.5$, a better estimate, but still an approximation. We can continue picking points closer and closer to (2,4) on the graph of f, and then calculating the slopes of the lines through each of these points and the point (2,4):

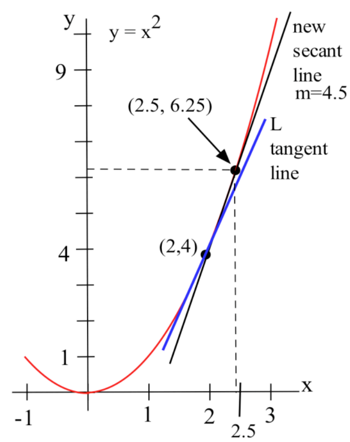

Fig. 2

Points to the left of (2,4)

x	$y = x^2$	slope of line through (x,y) and (2,4)
1.5	2.25	3.5
1.9	3.61	3.9
1.99	3.9601	3.99

Points to the right of (2,4)

x	$y = x^2$	slope of line through (x,y) and (2,4)
3	9	5
2.5	6.25	4.5
2.01	4.0401	4.01

The only thing special about the x–values we picked is that they are numbers which are close, and very close, to x = 2. Someone else might have picked other nearby values for x. As the points we pick get closer and closer to the point (2,4) on the graph of $y = x^2$, the slopes of the lines through the points and (2,4) are better approximations of the slope of the tangent line, and these slopes are getting closer and closer to 4.

Practice 1: What is the slope of the line through (2,4) and (x, y) for $y = x^2$ and x = 1.994? x = 2.0003?

We can bypass much of the calculating by not picking the points one at a time: let's look at a general point near (2,4). Define $x = 2 + h$ so h is the increment from 2 to x (Fig. 3). If h is small, then $x = 2 + h$ is close to 2 and the point $(2+h, f(2+h)) = (2+h, (2+h)^2)$ is close to (2,4). The slope m of the line through the points (2,4) and $(2+h, (2+h)^2)$ is a good approximation of the slope of the tangent line at the point (2,4):

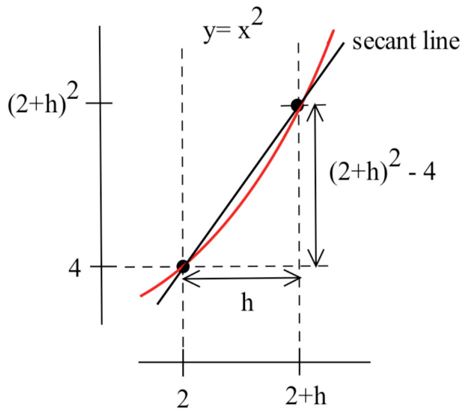

Fig. 3

$$m = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 4}{(2+h) - 2}$$

$$= \frac{\{4 + 4h + h^2\} - 4}{h} = \frac{4h + h^2}{h} = \frac{h(4 + h)}{h} = 4 + h .$$

If h is very small, then $m = 4 + h$ is a very good approximation to the slope of the tangent line, and $m = 4 + h$ is very close to the value 4. The value $m = 4 + h$ is called the slope of the **secant line** through the two points (2,4) and $(2+h, (2+h)^2)$. The limiting value 4 of $m = 4 + h$ as h gets smaller and smaller is called the **slope of the tangent line** to the graph of f at (2,4).

Example 1: Find the slope of the line tangent to $f(x) = x^2$ at the point (1,1) by evaluating the slope of the secant line through (1,1) and (1+h, f(1+h)) and then determining what happens as h gets very small (Fig. 4).

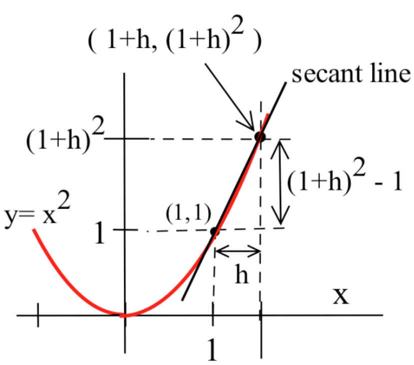

Fig. 4

Solution: The slope of the secant line through the points (1,1) and (1+h, f(1+h)) is

$$m = \frac{f(1+h) - 1}{(1+h) - 1} = \frac{(1+h)^2 - 1}{h} = \frac{\{ 1 + 2h + h^2 \} - 1}{h}$$

$$= \frac{2h + h^2}{h} = 2 + h.$$ As h gets very small, the value of m approaches the value 2, the slope of tangent line at the point (1,1).

Practice 2: Find the slope of the line tangent to the graph of $y = f(x) = x^2$ at the point (–1,1) by finding the slope of the secant line, m_{sec}, through the points (–1,1) and (–1+h, f(–1+h)) and then determining what happens to m_{sec} as h gets very small.

FALLING TOMATO

Suppose we drop a tomato from the top of a 100 foot building (Fig. 5) and time its fall .

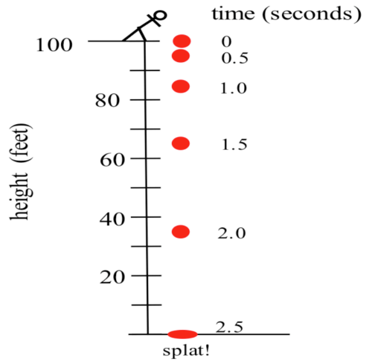

Fig. 5

time (s)	height (ft)
0	100
0.5	96
1.0	84
1.5	64
2.0	36
2.5	0

Some questions are easy to answer directly from the table:

(a) How long did it take for the tomato n to drop 100 feet? (2.5 seconds)

(b) How far did the tomato fall during the first second? (100 – 84 = 16 feet)

(c) How far did the tomato fall during the last second? (64 – 0 = 64 feet)

(d) How far did the tomato fall between t =.5 and t = 1? (96 – 84 = 12 feet)

Some other questions require a little calculation:

(e) What was the average velocity of the tomato during its fall?

$$\textbf{Average velocity} = \frac{\textbf{distance fallen}}{\textbf{total time}} = \frac{\boldsymbol{\Delta}\textbf{ position}}{\boldsymbol{\Delta}\textbf{ time}} = \frac{-100 \text{ ft}}{2.5 \text{ s}} = -40 \text{ ft/s} .$$

(f) What was the average velocity between t=1 and t=2 seconds?

$$\text{Average velocity} = \frac{\Delta \text{ position}}{\Delta \text{ time}} = \frac{36 \text{ ft} - 84 \text{ ft}}{2 \text{ s} - 1 \text{ s}} = \frac{-48 \text{ ft}}{1 \text{ s}} = -48 \text{ ft/s} .$$

Some questions are more difficult.

(g) How fast was the tomato falling 1 second after it was dropped?

This question is significantly different from the previous two questions about average velocity. Here we want the **instantaneous velocity,** the velocity at an instant in time. Unfortunately the tomato is not equipped with a speedometer so we will have to give an approximate answer.

One crude approximation of the instantaneous velocity after 1 second is simply the average velocity during the entire fall, –40 ft/s . But the tomato fell slowly at the beginning and rapidly near the end so the "–40 ft/s" estimate may or may not be a good answer.

We can get a better approximation of the instantaneous velocity at t=1 by calculating the average velocities over a short time interval near t = 1 . The average velocity between t = 0.5 and t = 1 is $\frac{-12 \text{ feet}}{0.5 \text{ s}} = -24$ ft/s, and the average velocity between t = 1 and t = 1.5 is $\frac{-20 \text{ feet}}{.5 \text{ s}} = -40$ ft/s so we can be reasonably sure that the instantaneous velocity is between –24 ft/s and –40 ft/s.

In general, the shorter the time interval over which we calculate the average velocity, the better the average velocity will approximate the instantaneous velocity. The average velocity over a time interval is $\frac{\Delta \text{position}}{\Delta \text{time}}$, which is the slope of the **secant line** through two points on the graph of height versus time (Fig. 6). The instantaneous velocity at a particular time and height is the slope of the **tangent line** to the graph at the point given by that time and height.

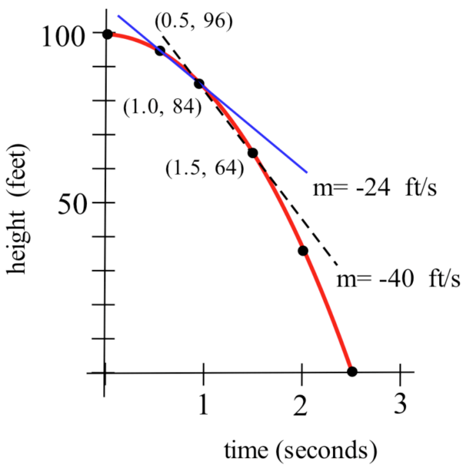

Fig. 6

Average velocity $= \frac{\Delta \textbf{position}}{\Delta \textbf{time}}$

= slope of the secant line through 2 points.

Instantaneous velocity = slope of the line tangent to the graph.

Practice 3: Estimate the velocity of the tomato 2 seconds after it was dropped.

GROWING BACTERIA

Suppose we set up a machine to count the number of bacteria growing on a petri plate (Fig. 7). At first there are few bacteria so the population grows slowly. Then there are more bacteria to divide so the population grows more quickly. Later, there are more bacteria and less room and nutrients available for the expanding population, so the population grows slowly again. Finally, the bacteria have used up most of the nutrients, and the population declines as bacteria die.

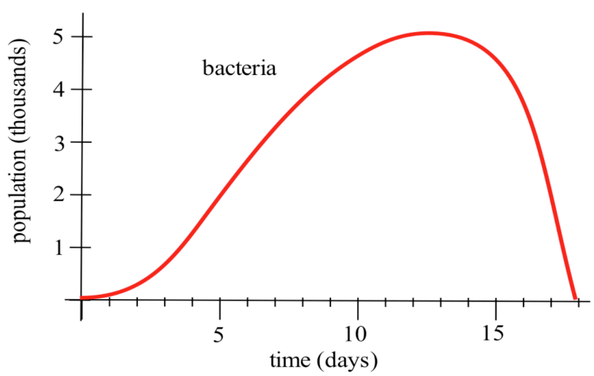

Fig. 7

The population graph can be used to answer a number of questions.

(a) What is the bacteria population at time t = 3 days?
(Answer: about 500 bacteria)

(b) What is the population increment from t = 3 to t = 10 days? (about 4000 bacteria)

(c) What is the **rate** of population growth from t = 3 to t = 10 days? (Fig. 7)

Solution: The rate of growth from t = 3 to t = 10 is the average change in population during that time:

$$\text{average change in population} = \frac{\text{change in population}}{\text{change in time}} = \frac{\Delta \text{population}}{\Delta \text{time}}$$

$$= \frac{4000 \text{ bacteria}}{7 \text{ days}} \approx 570 \text{ bacteria/day} .$$

This is the slope of the secant line through the two points (3, 500) and (10, 4500).

(d) What is the **rate** of population growth on the third day, at $t = 3$?

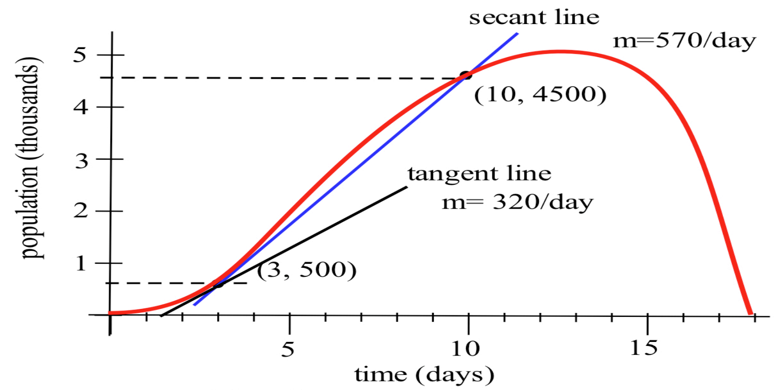

Fig. 8

Solution: This question is asking for the **instantaneous** rate of population change, the slope of the line which is **tangent** to the population curve at (3, 500). If we sketch a line approximately tangent to the curve at (3, 500) and pick two points near the ends of the tangent line segment (Fig. 8), we can estimate that instantaneous rate of population growth is approximately 320 bacteria/day .

Average population growth rate = $\dfrac{\Delta \text{ population}}{\Delta \text{ time}}$

= **slope of the secant line through 2 points.**

Instantaneous population growth rate = **slope of the line tangent to the graph.**

Practice 4: Approximately what was the average change in population between $t = 9$ and $t = 13$?
Approximately what was rate of population growth at $t = 9$ days?

The tangent line problem, the instantaneous velocity problem and the instantaneous growth rate problem are all similar. In each problem we wanted to know how rapidly something was **changing at an instant in time**, and each problem turned out to be finding the **slope of a tangent line**. The approach in each problem was also the same: find an approximate solution and then examine what happened to the approximate solution over shorter and shorter intervals. We will often use this approach of finding a limiting value, but before we can use it effectively we need to describe the concept of a limit with more precision.

PROBLEMS

1. What is the slope of the line through (3,9) and (x, y) for $y = x^2$ and $x = 2.97$? $x = 3.001$? $x = 3+h$? What happens to this last slope when h is very small (close to 0)? Sketch the graph of $y = x^2$ for x near 3.

2. What is the slope of the line through (–2,4) and (x, y) for $y = x^2$ and $x = -1.98$? $x = -2.03$? $x = -2+h$? What happens to this last slope when h is very small (close to 0)? Sketch the graph of $y = x^2$ for x near –2.

3. What is the slope of the line through (2,4) and (x, y) for $y = x^2 + x - 2$ and $x = 1.99$? $x = 2.004$? $x = 2+h$? What happens to this last slope when h is very small? Sketch the graph of $y = x^2 + x - 2$ for x near 2.

4. What is the slope of the line through (–1,–2) and (x, y) for $y = x^2 + x - 2$ and $x = -.98$? $x = -1.03$? $x = -1+h$? What happens to this last slope when h is very small? Sketch the graph of $y = x^2 + x - 2$ for x near –1.

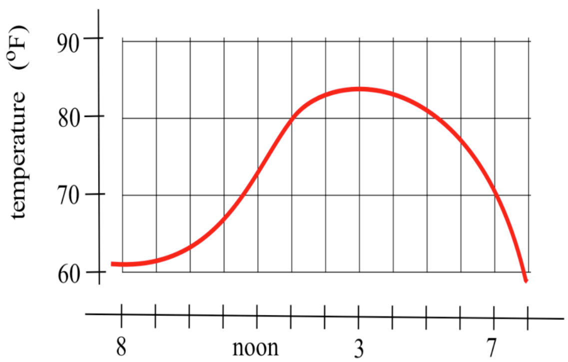

Fig. 9

5. Fig. 9 shows the temperature during a day in Ames.
(a) What was the average change in temperature from 9 am to 1 pm?
(b) Estimate how fast the temperature was rising **at** 10 am and **at** 7 pm?

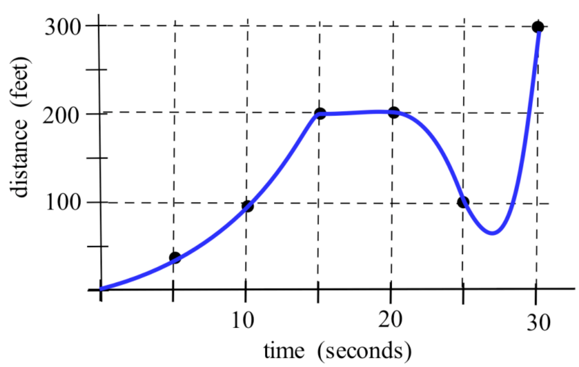

Fig. 10

6. Fig. 10 shows the distance of a car from a measuring position located on the edge of a straight road.
(a) What was the average velocity of the car from $t = 0$ to $t = 30$ seconds?
(b) What was the average velocity of the car from $t = 10$ to $t = 30$ seconds?
(c) About how fast was the car traveling **at** $t = 10$ seconds? **at** $t = 20$ s ? **at** $t = 30$ s ?
(d) What does the horizontal part of the graph between $t = 15$ and $t = 20$ seconds mean?
(e) What does the negative velocity at $t = 25$ represent?

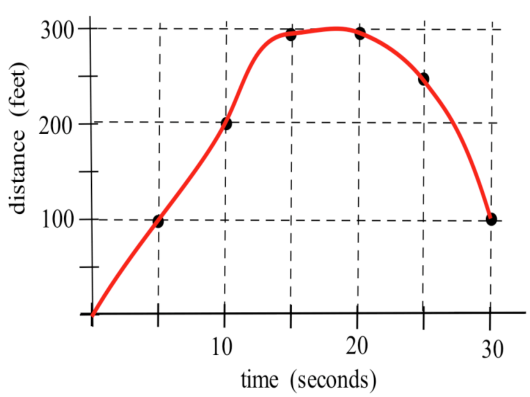

Fig. 11

7. Fig. 11 shows the distance of a car from a measuring position located on the edge of a straight road.
(a) What was the average velocity of the car from $t = 0$ to $t = 20$ seconds?
(b) What was the average velocity from $t = 10$ to $t = 30$ seconds?
(c) About how fast was the car traveling **at** $t = 10$ seconds? **at** $t = 20$ s ? **at** $t = 30$ s ?

8. Fig. 12 shows the composite developmental skill level of chessmasters at different ages as determined by their performance against other chessmasters. (From "Rating Systems for Human Abilities", by W.H. Batchelder and R.S. Simpson, 1988. UMAP Module 698.)

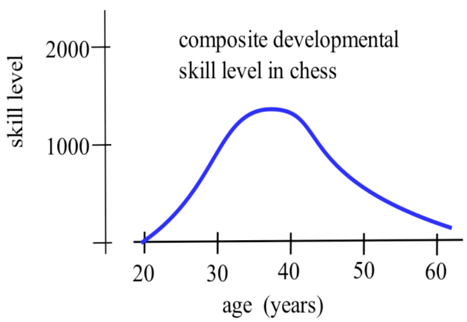

Fig. 12

(a) At what age is the "typical" chessmaster playing the best chess?

(b) At approximately what age is the chessmaster's skill level increasing most rapidly?

(c) Describe the development of the "typical" chessmaster's skill in words.

(d) Sketch graphs which you **think** would reasonably describe the performance levels versus age for an athlete, a classical pianist, a rock singer, a mathematician, and a professional in your major field.

Problems 9 and 10 define new functions A(x) in terms of AREAS bounded by the functions $y = 3$ and $y = x + 1$. This may seem a strange way to define a functions A(x), but this idea will become important later in calculus. We are just getting an early start here.

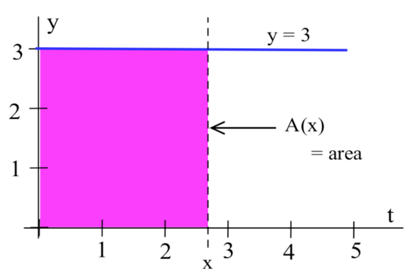

Fig. 13

9. Define A(x) to be the **area** bounded by the x and y axes, the horizontal line $y = 3$, and the vertical line at x (Fig. 13). For example, $A(4) = 12$ is the area of the 4 by 3 rectangle.
   a) Evaluate A(0), A(1), A(2), A(2.5) and A(3).
   b) What area would $A(4) - A(1)$ represent in the figure?
   c) Graph $y = A(x)$ for $0 \le x \le 4$.

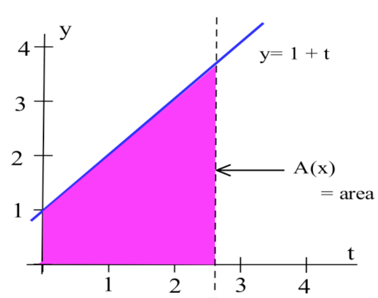

Fig. 14

10. Define A(x) to be the **area** bounded by the x and y axes, the line $y = x + 1$, and the vertical line at x (Fig. 14). For example, $A(4) = 12$.
    a) Evaluate A(0), A(1), A(2), A(2.5) and A(3).
    b) What area would $A(3) - A(1)$ represent in the figure?
    c) Graph $y = A(x)$ for $0 \le x \le 4$.

Section 1.0 **PRACTICE Answers**

Practice 1: $y = x^2$

If $x = 1.994$, then $y = 3.976036$ so the slope between $(2, 4)$ and (x, y) is

$$\frac{4 - y}{2 - x} = \frac{4 - 3.976036}{2 - 1.994} = \frac{0.023964}{0.006} \approx 3.994 .$$

If $x = 2.0003$, then $y \approx 4.0012$ so the slope between $(2, 4)$ and (x, y) is

$$\frac{4 - y}{2 - x} = \frac{4 - 4.0012}{2 - 2.0003} = \frac{-0.0012}{0.0003} \approx 4.0003 .$$

Practice 2: $m_{sec} = \frac{f(-1 + h) - (1)}{(-1 + h) - (-1)} = \frac{(-1 + h)^2 - 1}{h} = \frac{1 - 2h + h^2 - 1}{h} = \frac{h(-2 + h)}{h} = \mathbf{-2 + h}$

As $h \to 0$, $m_{sec} = -2 + h \to \mathbf{-2}$.

Practice 3: The average velocity between $t = 1.5$ and $t = 2.0$ is $\frac{36 - 64 \text{ feet}}{2.0 - 1.5 \text{ sec}} = -56$ feet per second.

The average velocity between $t = 2.0$ and $t = 2.5$ is $\frac{0 - 36 \text{ feet}}{2.5 - 2.0 \text{ sec}} = -72$ feet per second.

The velocity **at** $t = 2.0$ is somewhere between -56 ft/sec and -72 ft/sec, probably about the middle of this interval: $\frac{(-56) + (-72)}{2} = \mathbf{-64}$ **ft/sec**.

Practice 4: (a) When $t = 9$ days, the population is approximately $P = 4{,}200$ bacteria. When $t = 13$, $P \approx 5{,}000$. The average change in population is approximately

$$\frac{5000 - 4200 \text{ bacteria}}{13 - 9 \text{ days}} = \frac{800 \text{ bacteria}}{4 \text{ days}} = \mathbf{200 \text{ bacteria per day}}.$$

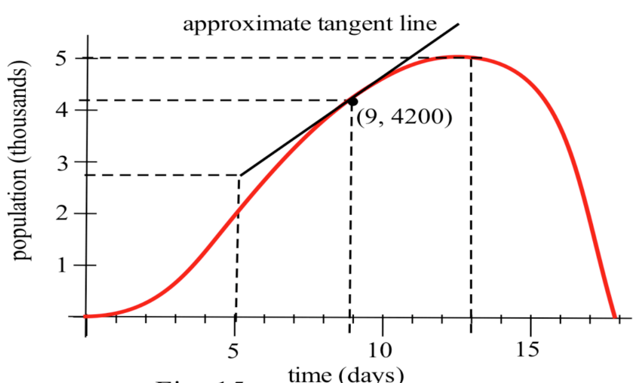

Fig. 15

(b) To find the rate of population growth at $t = 9$ days, sketch the line tangent to the population curve at the point $(9, 4200)$ and then use $(9, 4200)$ and another point on the tangent line to calculate the slope of the line. Using the approximate values $(5, 2800)$ and $(9, 4200)$, the slope of the tangent line at the point $(9, 4200)$ is approximately

$$\frac{4200 - 2800 \text{ bacteria}}{9 - 5 \text{ days}} = \frac{1400 \text{ bacteria}}{4 \text{ days}} \approx \mathbf{350 \text{ bacteria/day}} .$$