

12.3 ARC LENGTH AND CURVATURE OF SPACE CURVES

In earlier sections we have emphasized the dynamic nature of vector-valued functions by considering them as the path of a moving object. This is a very fruitful approach, but sometimes it is useful to consider a space curve as a static object and to investigate some of its geometric properties. This section considers two geometric aspects of space curves: arc length (how long is it along the curve from one point to another point?) and curvature (how quickly does the curve bend?).

Arc Length

In Section 9.4 we went through a careful derivation of an integral formula for finding the length of a parametric curve

$(x(t), y(t))$ from $t = a$ to $t = b$ (Fig. 1): we

- (1) partitioned the interval $[a, b]$ for the variable t and found the points $(x(t_i), y(t_i))$,
- (2) found the lengths of the line segments between consecutive points along the curve
- (3) added the lengths of the line segments (an approximation of the length of the curve) and got a Riemann sum
- (4) took the limit of the Riemann sum to an integral formula for the length of the curve

A very similar process also works for finding the length of a curve given by a vector-valued function in three dimensions, a space curve (Fig. 2), and we define the result of that process to be the length of a space curve.

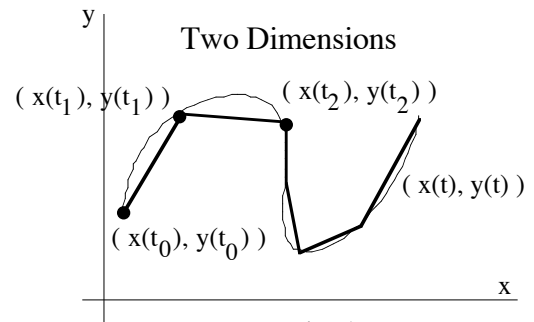


Fig. 1

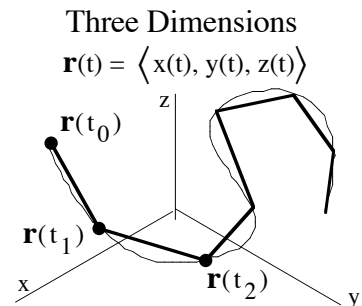


Fig. 2

Definition: **Distance Along the Graph of a Vector-Valued Function**
 (Arc Length of a Space Curve)

If $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ and $x'(t), y'(t), z'(t)$ are continuous
 then the **distance traveled, L , along the graph** of $\mathbf{r}(t)$ from $t = a$ to $t = b$ is

$$\text{distance traveled } L = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

If we travel along each part of the curve $\mathbf{r}(t)$ exactly once, then the arc length of the curve is the distance traveled:

$$\text{arc length} = \text{distance traveled } L = \int_{t=a}^{t=b} |\mathbf{r}'(t)| dt = \int_{t=a}^{t=b} |\mathbf{v}(t)| dt .$$

Example 1: Represent the length of the helix $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ from $t = 0$ to $t = 2\pi$. (Fig. 3)

Solution:
$$L = \int_{t=0}^{t=2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (1)^2} dt$$

$$= \int_{t=0}^{t=2\pi} \sqrt{\sin^2(t) + \cos^2(t) + 1} dt = \int_{t=0}^{t=2\pi} \sqrt{2} dt = 2\pi\sqrt{2} \approx 8.89 .$$

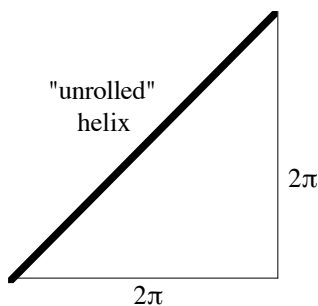
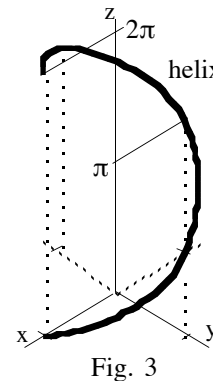


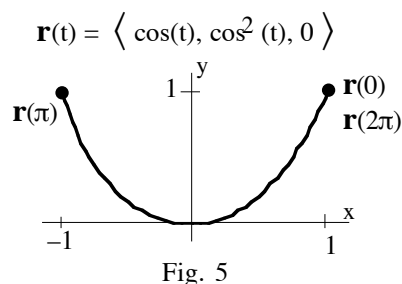
Fig. 4

(Actually, we can do this particular problem without calculus. If we "unroll" the helix (Fig. 4) we get a right triangle with base 2π (the circumference of the circle with radius 1) and height 2π (the value of $z(t)$ when $t = 2\pi$). The length of the helix is the length of the hypotenuse of this triangle: hypotenuse = $\sqrt{(2\pi)^2 + (2\pi)^2} = 2\pi\sqrt{2}$.)

Practice 1: Represent the length of the graph of $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ from $t = 0$ to $t = 2$ as an integral and use Simpson's Rule with $n = 20$ to approximate the value of the integral.

Example 2: Represent the length of the graph of $\mathbf{r}(t) = \langle \cos(t), \cos^2(t), 0 \rangle$ from $t = 0$ to $t = 2\pi$ as an integral and use Simpson's Rule with $n = 20$ to approximate the value of the integral.

Solution: The graph of $\mathbf{r}(t)$ is part of a parabola (Fig. 5), and the distance traveled along the parabola is



$$\begin{aligned} \text{distance} &= \int_{t=0}^{t=2\pi} \sqrt{(-\sin(t))^2 + (-2\cos(t)\sin(t))^2 + (0)^2} dt \\ &= \int_{t=0}^{t=2\pi} \sqrt{\sin^2(t) + 4\cos^2(t)\sin^2(t)} dt \approx 5.916 \quad (\text{using Simpson's Rule with } n=20). \end{aligned}$$

But in this example, the distance is NOT the length of the graph. As t goes from 0 to π we travel along the parabola from the point $(1, 1, 0)$ to the origin and on to $(-1, 1, 0)$. As t goes from π to 2π we travel back along the parabola to the starting point $(1, 1, 0)$. As t goes from 0

to 2π we cover the parabola twice so the length of the parabola is half of the distance traveled:

$$\text{length} = \frac{1}{2}(\text{distance travelled}) \approx \frac{1}{2}(5.916) = 2.958.$$

We could have calculated the length of the curve as the value of integral from $t = 0$ to $t = \pi$.

Parameterizing a Curve with respect to Arc Length

So far in our dealings with parametric space curves and vector-valued functions we have treated the curves of functions of the variable t and often thought about t as representing time. We have referred to the position vector $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ as representing the position $(x(t), y(t), z(t))$ of an object at time t . With a space curve, however, it is sometimes more useful to represent a location on the curve as a function of "how far along the curve" we are. For example, if we are giving someone directions to a good picnic spot in the mountains, we might describe the location (Fig. 6) as "drive 5.3 miles along the road from the turnoff" indicating that the driver should travel 5.3 miles from the beginning of the mountain road. This "how far along the road or curve" method avoids the obvious drawbacks of directions such as "drive 9 minutes at 35 miles per hour."



Fig. 6

Similarly, interstate highways are often marked with signs indicating how far we are from the the beginning of the road or the point where the road entered our state. It is usually more useful to describe the location of a knot in a wire as "17 inches from the end of the wire." That description does not depend on how fast we move along the wire or the orientation of the wire in space or even on the shape of the curve. The benefits of giving directions in terms of "how far along a road or wire" are the same for describing a location on a curve as "how far along the curve." The description of locations along a curve in terms of distance along the curve is called a **parameterization of the curve in terms of arc length**.

Definition: Arc Length Function $s(t)$

For a curve that begins at $\mathbf{r}(a) = \langle x(a), y(a), z(a) \rangle$ with continuous x', y' and z' , the distance along the curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ at time t is the arc length function $s(t)$ with

$$s(t) = \int_{u=a}^{u=t} \sqrt{\left(\frac{dx(u)}{du}\right)^2 + \left(\frac{dy(u)}{du}\right)^2 + \left(\frac{dz(u)}{du}\right)^2} du = \int_{u=a}^{u=t} |\mathbf{r}'(u)| du.$$

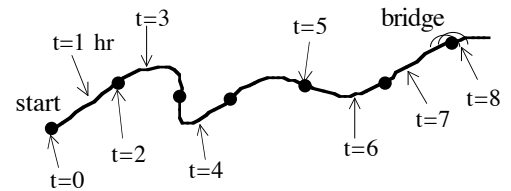


Fig. 7

Example 3: Fig. 7 illustrates the path \mathbf{r} of a salmon swimming

up a river marked with dots at 1 mile intervals.

- (a) Label the location of $\mathbf{r}(4)$ with an "X" for \mathbf{r} parameterized in terms of arc length.
- (b) Label the location of $\mathbf{r}(4)$ with an "O" for \mathbf{r} parameterized in terms of time.
- (c) For an arc length parameterization, find A so $\mathbf{r}(A) =$ bridge.
- (d) During which time interval was the fish swimming the fastest?

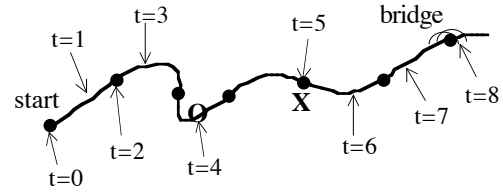


Fig. 8

- Solution:
- (a) and (b) Fig. 8 shows the correct locations of the "X" and the "O."
 - (c) The bridge is 6 miles from the beginning of the river so $A = 6$.
 - (d) The fish swims the greatest distance between $t = 4$ and $t = 5$, so it was swimming fastest during that 1 hour time interval.

Practice 2: For the salmon in Fig. 7

- (a) label the location of $\mathbf{r}(3)$ with an "S" for \mathbf{r} parameterized in terms of arc length.
- (b) Label the location of $\mathbf{r}(3)$ with an "T" for \mathbf{r} parameterized in terms of time.
- (c) For a time parameterization, find B so $\mathbf{r}(B) =$ bridge.
- (d) During which time interval was the fish swimming the slowest?

For most curves, it is difficult to find a simple formula for the arc length function $s(t)$. But sometimes we do get such a nice result for $s(t)$ that we can solve for $t(s)$, t in terms of s , and then we can rewrite the original parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ as $\mathbf{r}(t(s)) = \langle x(t(s)), y(t(s)), z(t(s)) \rangle$.

Example 4: Write an arc length parameterization of the helix $\mathbf{r}(t) = \langle 3t, 4\cos(t), 4\sin(t) \rangle$ using $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$ as the starting point.

Solution: $\mathbf{r}'(t) = \langle 3, -4\sin(t), 4\cos(t) \rangle$ for all t so

$$|\mathbf{r}'(t)| = \sqrt{(3)^2 + (-4\sin(t))^2 + (4\cos(t))^2} = \sqrt{9 + 16\sin^2(t) + 16\cos^2(t)} = 5 \text{ for all } t.$$

$$\text{Then } s = s(t) = \int_{u=0}^{u=t} |\mathbf{r}'(u)| \, du = \int_{u=0}^{u=t} 5 \, du = 5t \text{ so } t = \frac{s}{5}. \text{ By substituting } t(s) = \frac{s}{5} \text{ for } t,$$

the original parameterization $\mathbf{r}(t) = \langle 3t, 4\cos(t), 4\sin(t) \rangle$ becomes

$$\mathbf{r}(t(s)) = \langle 3t(s), 4\cos(t(s)), 4\sin(t(s)) \rangle = \langle 3\frac{s}{5}, 4\cos(\frac{s}{5}), 4\sin(\frac{s}{5}) \rangle, \text{ a function of } s \text{ alone.}$$

Practice 3: Write an arc length parameterization of the line $\mathbf{r}(t) = \langle 8t, t, 4t \rangle$ using $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$ as the starting point.

The conversions from "time parameterization" to "arc length parameterization" in the example and practice problems were relatively easy because the object was moving along each curve at a constant speed ($|\mathbf{r}'(t)|$ was constant). Usually this conversion is not that easy, but most of the time the arc length parameterization for a curve will be given so we will not need to translate to get from a "time parameterization" to an "arc length parameterization."

Curvature

Fig. 9 shows a space curve $\mathbf{r}(t)$ and unit tangent vectors $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ at several equally spaced (in terms of arc length) points along $\mathbf{r}(t)$. When the curve twists and bends sharply in the left part of the graph, the unit tangent vectors change direction rapidly from point to point. When the curve is almost straight and bends slowly, the unit tangent vectors also change direction slowly. This geometric pattern between the "bendedness" of a curve and the rate of change (with respect to arc length) of the direction of the unit tangent vectors leads to our definition of the **curvature** of a space curve at a point.

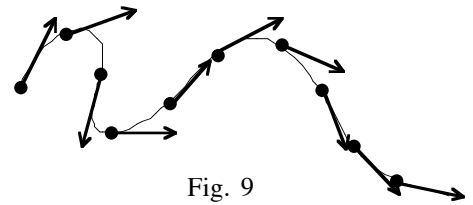


Fig. 9

Definition: Curvature

If \mathbf{r} is a space curve with unit tangent vector \mathbf{T} and arc length parameterization s , then the **curvature** of \mathbf{r} is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|. \quad (\kappa \text{ is the Greek letter "kappa"})$$

The curvature of a space curve is defined to be the magnitude of the rate of change of direction of the unit tangent vectors with respect to arc length. This definition of curvature is nicely motivated geometrically, but it is difficult to use for computations if we do not have an arc length parameterization of \mathbf{r} . However, the Chain Rule and the Fundamental Theorem of Calculus provide us with an easier way to actually calculate the curvature of a space curve $\mathbf{r}(t)$.

By the Chain Rule, $\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \cdot \frac{ds}{dt}$ so $\left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}/dt}{ds/dt} \right| = \left| \frac{\mathbf{T}'(t)}{ds/dt} \right|$. From the

Fundamental Theorem of Calculus and the definition of $s(t) = \int_{u=a}^{u=t} |\mathbf{r}'(u)| du$, we know that

$\frac{ds(t)}{dt} = |\mathbf{r}'(t)|$. By putting these two results together, we get a much easier to use formula for the

curvature of a space curve.

A Formula for Curvature: $\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$

Example 5: For a positive number A , the graph of $\mathbf{r}(t) = \langle A \cdot \cos(t), A \cdot \sin(t), 0 \rangle$ is a circle with radius A in the xy -plane. Find the curvature of this circle.

Solution: $\mathbf{r}'(t) = \langle -A \cdot \sin(t), A \cdot \cos(t), 0 \rangle$ so $|\mathbf{r}'(t)| = \sqrt{A^2 \sin^2(t) + A^2 \cos^2(t) + 0} = A$.

$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{r}'(t)}{A} = \langle -\sin(t), \cos(t), 0 \rangle$ so $\mathbf{T}'(t) = \langle -\cos(t), -\sin(t), 0 \rangle$ and $|\mathbf{T}'(t)| = 1$.

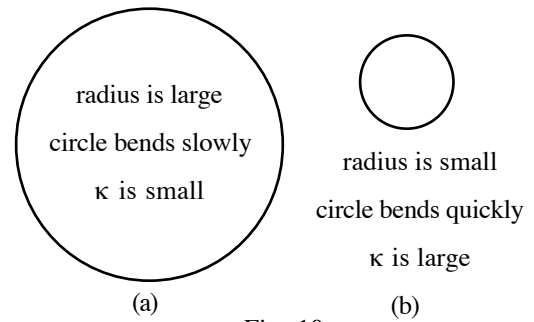


Fig. 10

Then, for all t , $\kappa = \left| \frac{\mathbf{T}'(t)}{\mathbf{r}'(t)} \right| = \frac{1}{A}$. The curvature of a circle of radius A is $\kappa = \frac{1}{A}$.

This agrees with our geometric idea of curvature:

- when the radius of the circle is large (Fig. 10a), the circle bends slowly and the curvature κ is small
- when the radius of the circle is small (Fig. 10b), the circle bends quickly and the curvature κ is large.

This pattern for curvature of circles leads to the definition of the radius of curvature of a curve.

Definition: The **radius of curvature** of $\mathbf{r}(t)$ is $\frac{1}{\text{curvature of } \mathbf{r}(t)} = \frac{1}{\kappa}$.

Practice 4: For A , B , and C not equal to 0, show that the line $\mathbf{r}(t) = \langle At, Bt, Ct \rangle$ has curvature $\kappa = 0$.

It was relatively straightforward to calculate the curvature in the example and practice problem because $|\mathbf{r}'(t)|$ was a constant. When $|\mathbf{r}'(t)|$ is not constant, it can be difficult to calculate $\mathbf{T}'(t)$, and some other formulas for curvature are often easier. The following formula for curvature looks complicated, but in practice it is often the easiest one to use.

"Easiest" Formula for Curvature: $\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$

A proof that this formula follows from the formula $\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$ is given in an appendix after the problem set.

Example 6: Use the formula $\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$ to determine the curvature and the radius of curvature of $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ when $t = 0, 1$, and 2 .

Solution: $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$ so $|\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4}$ and $\mathbf{r}''(t) = \langle 0, 2, 6t \rangle$.

$$\text{Then } \mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix}$$

$$= (6t^2)\mathbf{i} - (6t)\mathbf{j} + (2)\mathbf{k} \quad \text{and}$$

$$|\mathbf{r}' \times \mathbf{r}''| = \sqrt{(6t^2)^2 + (-6t)^2 + (2)^2} = \sqrt{36t^4 + 36t^2 + 4}.$$

$$\text{Putting this all together, we have } \kappa = \frac{\sqrt{36t^4 + 36t^2 + 4}}{(1 + 4t^2 + 9t^4)^{3/2}}.$$

$$\text{When } t = 0, \kappa = \frac{\sqrt{4}}{1} = 2 \quad \text{so the radius of curvature is } \frac{1}{2}.$$

$$\text{When } t = 1, \kappa = \frac{\sqrt{76}}{(14)^{3/2}} \approx 0.166 \quad \text{so the radius of curvature is approximately } \frac{1}{0.166} \approx 6.02.$$

$$\text{When } t = 2, \kappa = \frac{\sqrt{724}}{(161)^{3/2}} \approx 0.013 \quad \text{so the radius of curvature is approximately } 76.9.$$

For $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, as t grows larger (and is positive), the curvature κ becomes smaller.

Practice 5: Use the formula $\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$ to determine the curvature of

$$\mathbf{r}(t) = \langle t, \sin(t), 0 \rangle \text{ when } t = 0, \pi/4, \text{ and } \pi/2.$$

Curvature in Two Dimensions: $\mathbf{r}(t) = \langle x(t), y(t), 0 \rangle$ and $y = f(x)$

Every curve confined to the xy -plane can be thought of as a curve in space whose z -coordinate is always 0, and that approach leads to alternate formulas for the curvature of the graph.

If the curve we are dealing with is given parametrically in two dimensions as $(x(t), y(t))$ then we can consider it as the vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), 0 \rangle$ in three dimensions, and the curvature formula $\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$ "simplifies" to the following.

$$\text{The graph of } (x(t), y(t)) \text{ has curvature } \kappa = \frac{|x'y'' - x''y'|}{((x')^2 + (y')^2)^{3/2}}$$

where the derivatives of x and y are with respect to t .

If $y = f(x)$, then the curve can be parameterized in two dimensions using $x(t) = t$ and $y(t) = f(x) = f(t)$.

With this parameterization $\mathbf{r}(t) = \langle x(t), y(t), 0 \rangle = \langle t, y(t), 0 \rangle$ and we have $x'(t) = 1$ and $x''(t) = 0$ so the previous pattern reduces to

$$\kappa = \frac{|y''|}{(1 + (y')^2)^{3/2}} \text{ where the derivatives are with respect to } x.$$

These last two formulas for curvature are typically easier to use than the previous ones, but **they are only valid for two-dimensional graphs.**

Example 7: Use the appropriate formula to determine the curvature of $y = x^2$ when $x = 0, 1$ and 2 .

Solution: We can use the formula $\kappa = \frac{|y''|}{(1 + (y')^2)^{3/2}}$ with $y' = 2x$ and $y'' = 2$. Then

$$\kappa = \frac{|2|}{(1 + (2x)^2)^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}.$$

When $x = 0$, $\kappa = \frac{2}{(1 + 4x^2)^{3/2}} = \frac{2}{(1)^{3/2}} = 2$. When $x = 1$, $\kappa = \frac{2}{(1 + 4x^2)^{3/2}} = \frac{2}{(5)^{3/2}} \approx 0.179$.

When $x = 2$, $\kappa = \frac{2}{(1 + 4x^2)^{3/2}} = \frac{2}{(17)^{3/2}} \approx 0.029$.

Practice 6: Use the appropriate 2-dimensional formula to determine the curvature of $y = \sin(x)$ when $x = 0, 1$ and 2 . (Your answers should agree with your answers to Practice 5.)

PROBLEMS

In problems 1 – 4, determine the length of the helices for $0 \leq t \leq 2\pi$.

1. $\mathbf{r}(t) = \langle 2 \cdot \cos(t), 2 \cdot \sin(t), t \rangle$

2. $\mathbf{r}(t) = \langle 3 \cdot \cos(t), 3 \cdot \sin(t), t \rangle$

3. $\mathbf{r}(t) = \langle 4 \cdot \cos(t), 4 \cdot \sin(t), t \rangle$

4. $\mathbf{r}(t) = \langle R \cdot \cos(t), R \cdot \sin(t), t \rangle$

In problems 5 – 12, determine the length of the "modified helices" for $0 \leq t \leq 2\pi$. If necessary, use Simpson's Rule with $n = 20$ to approximate the arc length integrals.

5. $\mathbf{r}(t) = \langle 2 \cdot \cos(t), 3 \cdot \sin(t), t \rangle$

6. $\mathbf{r}(t) = \langle 2 \cdot \cos(t), 5 \cdot \sin(t), t \rangle$

7. $\mathbf{r}(t) = \langle A \cdot \cos(t), B \cdot \sin(t), t \rangle$

8. $\mathbf{r}(t) = \langle \cos(2t), \sin(2t), t \rangle$

9. $\mathbf{r}(t) = \langle t \cdot \cos(t), t \cdot \sin(t), t \rangle$

10. $\mathbf{r}(t) = \langle 2t \cdot \cos(t), t \cdot \sin(t), t \rangle$

11. $\mathbf{r}(t) = \langle 2t \cdot \cos(t), t \cdot \sin(t), t \rangle$

12. $\mathbf{r}(t) = \langle t^2 \cdot \cos(t), t^2 \cdot \sin(t), t \rangle$

In problems 13 – 16, determine the length of the Bezier curves for $0 \leq t \leq 1$. If necessary, use Simpson's Rule with $n = 20$ to approximate the arc length integrals.

13. $\mathbf{r}(t) = \langle 3(1-t)^2t + 3(1-t)t^2, 9(1-t)t^2 + 2t^3, (1-t)^3 + 6(1-t)^2t \rangle$

14. $\mathbf{r}(t) = \langle 2(1-t)^3 + 6(1-t)t^2, 3(1-t)^2t + 3(1-t)t^2 + 3t^3, 3(1-t)^2t + 3(1-t)t^2 \rangle$

15. The Bezier curve determined by the control points $P_0 = (2, 0, 0)$, $P_1 = (0, 1, 1)$, $P_2 = (2, 1, 1)$, and $P_3 = (0, 3, 0)$.

16. The Bezier curve determined by the control points $P_0 = (2, 0, 0)$, $P_1 = (0, 3, 2)$, $P_2 = (2, 0, 3)$, and $P_3 = (0, 3, 0)$.

In problems 17 – 22, determine the curvature of the given curves at the specified points.

17. $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ when $t = 0, \pi/4$, and $\pi/2$.

18. $\mathbf{r}(t) = \langle 3 \cdot \cos(t), 3 \cdot \sin(t), t \rangle$ when $t = 0, \pi/4$, and $\pi/2$.

19. $\mathbf{r}(t) = \langle R \cdot \cos(t), R \cdot \sin(t), t \rangle$ when $t = 0, \pi/4$, and $\pi/2$.

20. $\mathbf{r}(t) = \langle 5 + 3t, 2 - t, 3 - 2t \rangle$ when $t = 0, 2$, and 7 .

21. $\mathbf{r}(t) = \langle 3(1-t)^2t + 3(1-t)t^2, 9(1-t)t^2 + 2t^3, (1-t)^3 + 6(1-t)^2t \rangle$ when $t = 0.2$ and 0.5 .

22. $\mathbf{r}(t) = \langle 2(1-t)^3 + 6(1-t)t^2, 3(1-t)^2t + 3(1-t)t^2 + 3t^3, 3(1-t)^2t + 3(1-t)t^2 \rangle$ when $t = 0.2$ and 0.5 .

In problems 23 – 26, determine the curvature and the radius of curvature of the given curves at the specified points.

23. $\mathbf{r}(t) = \langle 3 \cdot \cos(t), 5 \cdot \sin(t) \rangle$ when $t = 0, \pi/4,$ and $\pi/2$.

24. $\mathbf{r}(t) = \langle 2 \cdot \cos(t), 7 \cdot \sin(t) \rangle$ when $t = 0, \pi/4,$ and $\pi/2$.

25. $\mathbf{r}(t) = \langle A \cdot \cos(t), B \cdot \sin(t) \rangle$ when $t = 0, \pi/4,$ and $\pi/2$.

26. $\mathbf{r}(t) = \langle t \cdot \cos(t), t \cdot \sin(t) \rangle$ when $t = 1, 2,$ and 3 .

27. Determine the curvature of $y = 3x + 5$ when $x = 1, 2,$ and 3 . For what value of x is the curvature of $y = 3x + 5$ maximum?

28. Determine the curvature of $y = Ax + B$ when $x = 1, 2,$ and 3 . For what value of x is the curvature of $y = 3x + 5$ maximum?

29. Determine the curvature of $y = x^2$ when $x = 1, 2,$ and 3 . For what value of x is the curvature of $y = x^2$ maximum? For what value of x is the radius of curvature of $y = x^2$ minimum?

30. Determine the curvature of $y = x^3 - x$ when $x = 0, 1,$ and 2 . For what value of x is the curvature of $y = x^3 - x$ maximum? For what value of x is the radius of curvature of $y = x^3 - x$ minimum?

Practice Answers

Practice 1: $|\mathbf{r}'(t)| = \sqrt{1^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}$. Then

$$L = \int_{t=0}^{t=2} |\mathbf{r}'(t)| dt = \int_{t=0}^{t=2} \sqrt{1 + 4t^2 + 9t^4} dt \approx 9.57 \text{ (using Simpson's Rule with } n = 20\text{).}$$

Practice 2: (a) and (b) Fig. 11 shows the correct locations of the "S" and the "T."

(c) The bridge is 8 hours from the beginning of the river so $B = 8$.

(d) The fish swims the smallest distance between $t = 2$ and $t = 3$, so it was swimming slowest during that 1 hour time interval.

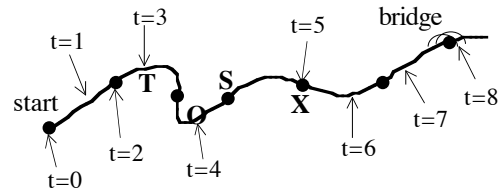


Fig. for Pr. 2

Practice 3: $|\mathbf{r}'(t)| = \sqrt{(8)^2 + (1)^2 + (4)^2} = 9$ for all t .

$$\text{Then } s = s(t) = \int_{u=0}^{u=t} |\mathbf{r}'(u)| du = \int_{u=0}^{u=t} 9 du = 9t \text{ so } t = \frac{s}{9}. \text{ By substituting } t(s) = \frac{s}{9} \text{ for } t,$$

the original parameterization $\mathbf{r}(t) = \langle 8t, t, 4t \rangle$ becomes

$$\mathbf{r}(t(s)) = \langle 8t(s), t(s), 4t(s) \rangle = \langle 8\frac{s}{9}, \frac{s}{9}, 4\frac{s}{9} \rangle, \text{ a function of } s \text{ alone.}$$

Practice 4: $\mathbf{r}'(t) = \langle A, B, C \rangle$ so $|\mathbf{r}'(t)| = \sqrt{A^2 + B^2 + C^2}$.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{A^2 + B^2 + C^2}} \langle A, B, C \rangle \text{ which is a constant vector so } \mathbf{T}'(t) = \mathbf{0}.$$

$$\text{Then, for all } t, \kappa = \left| \frac{\mathbf{T}'(t)}{|\mathbf{r}'(t)|} \right| = \frac{0}{\sqrt{A^2 + B^2 + C^2}} = 0.$$

The curvature of a the line $\mathbf{r}(t) = \langle At, Bt, Ct \rangle$ is $\kappa = 0$.

Practice 5: $\mathbf{r}'(t) = \langle 1, \cos(t), 0 \rangle$ so $|\mathbf{r}'(t)| = \sqrt{1 + \cos^2(t)}$ and $\mathbf{r}''(t) = \langle 0, -\sin(t), 0 \rangle$.

$$\text{Then } \mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \cos(t) & 0 \\ 0 & -\sin(t) & 0 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + (-\sin(t))\mathbf{k} \text{ and } |\mathbf{r}' \times \mathbf{r}''| = |-\sin(t)|.$$

$$\text{Putting this together, we have } \kappa = \frac{|-\sin(t)|}{(1 + \cos^2(t))^{3/2}}.$$

$$\text{When } t=0, \kappa = \frac{0}{(1+1)^{3/2}} = 0. \quad \text{When } t=\pi/4, \kappa = \frac{\sqrt{2}/2}{(1+1/2)^{3/2}} \approx 0.385.$$

$$\text{When } t=\pi/2, \kappa = \frac{1}{(1+0)^{3/2}} = 1$$

Practice 6: $y = \sin(x)$, $y'(x) = \cos(x)$, and $y''(x) = -\sin(x)$. Then

$$\kappa = \frac{|y''|}{(1+(y')^2)^{3/2}} = \frac{|-\sin(x)|}{(1+\cos^2(x))^{3/2}} \quad \text{which is the same result we got in Practice 5.}$$

Appendix: A proof that $\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$

Since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ and $|\mathbf{r}'(t)| = \frac{ds}{dt}$ we know that $\mathbf{r}'(t) = |\mathbf{r}'(t)| \mathbf{T}(t) = \frac{ds}{dt} \mathbf{T}(t)$.

Then, by the Product rule for Derivatives, $\mathbf{r}''(t) = \frac{d}{dt} \left\{ \frac{ds}{dt} \mathbf{T}(t) \right\} = \frac{d^2s}{dt^2} \mathbf{T}(t) + \frac{ds}{dt} \mathbf{T}'(t)$.

Replacing \mathbf{r}' and \mathbf{r}'' with these results and using the distributive pattern $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$ for the cross product, we have

$$\mathbf{r}' \times \mathbf{r}'' = \frac{ds}{dt} \mathbf{T}(t) \times \left\{ \frac{d^2s}{dt^2} \mathbf{T} + \frac{ds}{dt} \mathbf{T}' \right\} = \frac{ds}{dt} \frac{d^2s}{dt^2} \{ \mathbf{T} \times \mathbf{T} \} + \left\{ \frac{ds}{dt} \right\}^2 \{ \mathbf{T} \times \mathbf{T}' \}.$$

We know for every vector \mathbf{V} that $\mathbf{V} \times \mathbf{V} = \mathbf{0}$ (the zero vector) so $\mathbf{T} \times \mathbf{T} = \mathbf{0}$.

We also know that $|\mathbf{T}(t)| = 1$ for all t so from Example 3 of Section 12.2 we can conclude that \mathbf{T} is perpendicular to \mathbf{T}' . Then $|\mathbf{T} \times \mathbf{T}'| = |\mathbf{T}| |\mathbf{T}'| |\sin(\theta)| = |\mathbf{T}| |\mathbf{T}'| = |\mathbf{T}'|$. Using these results together with the previous result for $\mathbf{r}' \times \mathbf{r}''$ we have

$$|\mathbf{r}' \times \mathbf{r}''| = \left\{ \frac{ds}{dt} \right\}^2 |\mathbf{T}'| = |\mathbf{r}'|^2 |\mathbf{T}'| \quad \text{so } |\mathbf{T}'| = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^2}.$$

Finally, $\frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$, the result we wanted.

