7.1 INVERSE FUNCTIONS

One-to-one functions are important because their equations, \( f(x) = k \), have (at most) a single solution.

One-to-one functions are also important because they are the functions that can be uniquely "undone:" if \( f \) is a one-to-one function, then there is another function \( g \) which "undoes" the effect of \( f \) so \( g(f(x)) = x \), the original input (Fig. 1). The function \( g \) which "undoes" the effect of \( f \) is called the inverse function of \( f \) or simply the inverse of \( f \). If \( f \) is a function for encoding a message, then the inverse of \( f \) is the function that decodes an encoded message to get back to the original message. The functions \( e^x \) and \( \ln(x) \) "undo" the effects of each other: \( \ln(e^x) = x \) and \( e^{\ln(x)} = x \) (for \( x > 0 \)). The functions \( e^x \) and \( \ln(x) \) are inverses of each other.

In this section we examine some of the properties of inverse functions and see how to find the inverse of a function which is given by a table of data, a graph, or a formula.

Definition: If \( f \) and \( g \) are functions which satisfy \( g(f(x)) = x \) and \( f(g(x)) = x \), then \( g \) is the inverse of \( f \), \( f \) is the inverse of \( g \), and \( f \) and \( g \) are a pair of inverse functions.

The inverse function of \( f \) is written \( f^{-1} \) (pronounced "eff inverse").

(Be careful. \( f^{-1} \) does not mean \( 1/f \).)

**Example 1:** A function \( f \) is given by Table 1. Evaluate \( f^{-1}(0) \) and \( f^{-1}(1) \).

Solution: For every \( x \), \( f^{-1}(f(x)) = x \) so the value of \( f^{-1}(0) \) is the solution \( x \) of the equation \( f(x) = 0 \). The solution we want is \( x = 2 \), and we can check that \( f^{-1}(0) = f^{-1}(f(2)) = 2 \).

The value of \( f^{-1}(1) \) is the solution of the equation \( f(x) = 1 \), and from the table, we can see that solution is \( x = 3 \). We can check that \( f^{-1}(1) = f^{-1}(f(3)) = 3 \).

Similarly, \( f^{-1}(2) = 1 \) and \( f^{-1}(3) = 0 \). These results are shown in Table 2.

You should notice that if the ordered pair \( (a,b) \) is in the table for \( f \), then the reversed pair \( (b,a) \) is in the table for \( f^{-1} \).

**Practice 1:** The function \( g \) is given by Table 3. Write a table of values for \( g^{-1} \).

The method of interchanging the coordinates of a point on the graph (or in the table of values) of \( f \) to get a point on the graph (or in the table of values) of \( f^{-1} \) provides an efficient way of graphing \( f^{-1} \).
Theorem: If the point \((a,b)\) is on the graph of \(f\), then the point \((b,a)\) is on the graph of \(f^{-1}\).

Proof: If \((a,b)\) is on the graph of \(f\), then \(b = f(a)\). Since \(b\) and \(f(a)\) are equal, \(f^{-1}(b) = f^{-1}(f(a)) = a\) (by the definition of inverse functions) so the point \((b,a)\) is on the graph of \(f^{-1}\).

Graphically, when the coordinates of a point \((a,b)\) are interchanged to get a new point \((b,a)\), the new point is symmetric about the line \(y = x\) to the original point (Fig. 2). If you put a spot of wet ink at the point \((a,b)\) on a piece of paper and fold the paper along the line \(y = x\), there will be new spot of ink at the point \((b,a)\). Fig. 3 illustrates another graphical method for finding the location of the point \((b,a)\).

Corollary: The graphs of \(f\) and \(f^{-1}\) are symmetric about the line \(y = x\).

Example 2: The graph of \(f\) is shown in Fig. 4. Sketch the graph of \(f^{-1}\). 

Solution: Imagine the graph of \(f\) is drawn with wet ink, and fold the xy–plane along the line \(y = x\). When the plane is unfolded, the new graph is \(f^{-1}\). (Fig. 5)

Another method proceeds point–by–point. Pick several points \((a,b)\) on the graph of \(f\) and plot the symmetric points \((b,a)\). Use the new \((b,a)\) points as a guide for sketching the graph of \(f^{-1}\).
Practice 2: The graph of \( g \) is shown in Fig. 6. Sketch the graph of \( g^{-1} \). What happens to points on the graph of \( g \) that lie on the line \( y = x \), points of the form (\( a, a \))? 

Finding A Formula for \( f^{-1}(x) \)

When \( f \) is given by a table of values, a table of values for \( f^{-1} \) can be made by interchanging the values of \( x \) and \( y \) in the table for \( f \) as in Example 1. When \( f \) is given by a graph, a graph of \( f^{-1} \) can be made by reflecting the graph of \( f \) about the line \( y = x \) as in Example 2. When \( f \) is given by a formula, we can try to find a formula for \( f^{-1} \).

Example 3: The steps for wrapping a gift are (1) put gift in box, (2) put on paper, and (3) put on ribbon. What are the steps for opening the gift, the inverse of the wrapping operation?

Solution: (i) remove ribbon (undo step 3), (ii) remove paper (undo step 2), and (iii) remove gift from box (undo step 1). (Then show happiness and say thank you.) The point of this trivial example is to point out that the first unwrapping step undoes the last wrapping step, the second unwrapping step undoes the second-to-last wrapping step, . . . , and the last unwrapping step undoes the first wrapping step. This pattern holds for more numerical functions and their inverses too.

Example 4: The steps to evaluate \( f(x) = 9x + 6 \) are (1) multiply by 9 and (2) add 6. Write the steps, in words, for the inverse of this function, and then translate the verbal steps for the inverse into a formula for the inverse function.

Solution: (i) subtract 6 (undo step 2) and (ii) divide by 9 (undo step 1). \( f^{-1}(x) = \frac{x - 6}{9} \).

The following algorithm is a recipe for finding a formula for the inverse function.

Finding A Formula For \( f^{-1} \):

Start with a formula for \( f \): \( y = f(x) \).

**Interchange the roles of \( x \) and \( y \):** \( x = f(y) \).

**Solve** \( x = f(y) \) **for** \( y \).

The resulting formula for \( y \) is the inverse of \( f \): \( y = f^{-1}(x) \).

The "interchange" and "solve" steps in the algorithm effectively undo the original operations in reverse order.

Example 5: Find formulas for the inverses of \( f(x) = \frac{7x - 5}{4} \) and \( g(x) = 2e^{5x} \).
Solution: Starting with $y = f(x) = \frac{7x - 5}{4}$ and interchanging the roles of $x$ and $y$, we have $x = \frac{7y - 5}{4}$.

Solving for $y$, we get $4x = 7y - 5$, so $7y = 4x + 5$ and, finally, $y = \frac{4x + 5}{7} = f^{-1}(x)$.

Starting with $y = g(x) = 2e^{5x}$ and interchanging the roles of $x$ and $y$, we have $x = g(y) = 2e^{5y}$.

Solving for $y$, we get $\frac{x}{2} = e^{5y}$, so $\ln(\frac{x}{2}) = \ln(e^{5y}) = 5y$, and $y = \frac{1}{5} \cdot \ln(\frac{x}{2}) = g^{-1}(x)$.

**Practice 3:** Find formulas for the inverses of $f(x) = 2x - 5$, $g(x) = \frac{2x - 1}{x + 7}$, and $h(x) = 2 + \ln(3x)$.

Sometimes it is easy to "solve $x = f(y)$ for $y$," but not always. When we try to find a formula for the inverse of $y = f(x) = x + e^x$, the first step is easy: starting with $y = x + e^x$ and interchanging the roles of $x$ and $y$, we have $x = y + e^y$. Then, unfortunately, there is no way to algebraically solve the equation $x = y + e^y$ explicitly for $y$. The function $y = x + e^x$ has an inverse function, but we cannot find an explicit formula for the inverse.

**Which Functions Have Inverse Functions?**

We have seen how to find the inverse function for some functions given by tables of values, by graphs, and by formulas, but there are functions which do not have inverse functions. The only way a graph and its reflection about the line $y = x$ can both be function graphs ($f$ and $f^{-1}$ are both functions) is if the graph satisfies both the Vertical Line Test (so $f$ is a function) and the Horizontal Line Test (so $f^{-1}$ is a function). This is stated more formally in the next theorem, although we do not prove it.

**Theorem:** The function $f$ has an inverse function if and only if $f$ is one-to-one.

Two useful corollaries follow from this theorem.

**Corollary 1:** If $f$ is strictly increasing or is strictly decreasing, then $f$ has an inverse function.

**Corollary 2:** If $f'(x) > 0$ for all $x$ or $f'(x) < 0$ for all $x$, then $f$ has an inverse function.
7.1 Inverse Functions

Inverse Functions

When a function \( f \) has an inverse, the symmetry of the graphs of \( f \) and \( f^{-1} \) also gives information about slopes and derivatives.

**Example 6:** Suppose the points \( P = (1,2) \) and \( Q = (3,6) \) are on the graph of \( f \).

(a) Sketch the line through \( P \) and \( Q \), and find the slope of the line through \( P \) and \( Q \).

(b) Find the reflected points \( P^* \) and \( Q^* \) on the graph of \( f^{-1} \), sketch the line segment through \( P^* \) and \( Q^* \), and find the slope of the line through \( P^* \) and \( Q^* \).

**Solution:**

(a) The slope through \( P \) and \( Q \) is \( m = \frac{6 - 2}{3 - 1} = 2 \). The graphs are shown in Fig. 7.

(b) The reflected points, obtained by interchanging the first and second coordinates of each point on \( f \), are \( P^* = (2,1) \) and \( Q^* = (6,3) \). The slope of the line though \( P^* \) and \( Q^* \) is

\[
\frac{3 - 1}{6 - 2} = \frac{1}{2} = \text{slope of the segment through } P \text{ and } Q .
\]

In general, if \( P = (a,b) \) and \( Q = (x,y) \) are points on the graph of \( f \) (Fig. 8), then the reflected points \( P^* = (b,a) \) and \( Q^* = (y,x) \) are on the graph of \( f^{-1} \), and

\[
\{ \text{slope of the segment } P^*Q^* \} = \frac{1}{\text{slope of the segment } PQ} .
\]

Since the slope of a tangent line is the limit of slopes of secant lines, there is a similar relationship between the slope of the tangent line to \( f \) at the point \( (a,b) \) and slope of the tangent line to \( f^{-1} \) at the point \( (b,a) \). In Fig. 9, let the point \( Q^* \) approach the point \( P^* \) along the graph of \( f^{-1} \). Then

\[
(f^{-1})'(b) = \lim_{Q^* \to P^*} \{ \text{slope of the segment } P^*Q^* \} = \frac{1}{f'(a)} .
\]

**Derivative of the Inverse Function**

If \( b = f(a) \), and \( f \) is differentiable at the point \( (a,b) \), and \( f'(a) \neq 0 \), then \( a = f^{-1}(b) \), and \( f^{-1} \) is differentiable at the point \( (b,a) \).
Example 7: The point \((3, 1.1)\) is on the graph of \(f(x) = \ln(x)\) and \(f'(3) = 1/3\). Let \(g\) be the inverse function of \(f\). Give one point on the graph of \(g\) and evaluate \(g'\) at that point.

Solution: The point \((1.1, 3)\) is on the graph of \(g\), and \(g'(1.1) = \frac{1}{f'(3)} = \frac{1}{1/3} = 3\). In fact, the inverse of the natural logarithm is the exponential function \(e^x\), and we can check that \(e^{1.1} \approx 3\).

Example 8: In Table 4, fill in the values of \(f^{-1}(x)\) and \((f^{-1})'(x)\) for \(x = 0\) and 1.

Solution: \(f(3) = 0\) so \(f^{-1}(0) = 3\). \((f^{-1})'(0) = \frac{1}{f'(3)} = \frac{1}{2}\).

\[
\begin{array}{|c|c|c|c|c|}
\hline
x & f(x) & f'(x) & f^{-1}(x) & (f^{-1})'(x) \\
\hline
0 & 2 & 3 & & \\
1 & 3 & -2 & & \\
3 & 0 & 2 & & \\
\hline
\end{array}
\]

Table 4

Example 8: In Table 4, fill in the values of \(f^{-1}(x)\) and \((f^{-1})'(x)\) for \(x = 2\) and 3.

Practice 4: In Table 4, fill in the values of \(f^{-1}(x)\) and \((f^{-1})'(x)\) for \(x = 2\) and 3.

PROBLEMS

1. The values of \(f\) and \(f'\) are given in Table 5. Complete the columns for \(f^{-1}\) and \((f^{-1})'\).

2. The values of \(g\) and \(g'\) are given in Table 6. Complete the columns for \(g^{-1}\) and \((g^{-1})'\).

3. The values of \(h\) and \(h'\) are given in Table 7. Complete the columns for \(h^{-1}\) and \((h^{-1})'\).

4. The values of \(w\) and \(w'\) are given in Table 8. Complete the columns for \(w^{-1}\) and \((w^{-1})'\).

5. Fig. 10 shows the graph of \(f\). Sketch the graph of \(f^{-1}\).

6. Fig. 11 shows the graph of \(g\). Sketch
7.1 Inverse Functions

the graph of \( g^{-1} \).

7. If the graphs of \( f \) and \( f^{-1} \) intersect at the point \((a,b)\), how are \( a \) and \( b \) related?

8. If the graph \( f \) intersects the line \( y = x \) at \( x = a \), does the graph of \( f^{-1} \) intersect \( y = x \)? If so, where?

9. The steps to evaluate the function \( f(x) = \frac{7x - 5}{4} \) are (1) multiply by 7, (2) subtract 5, and (3) divide by 4. Write the steps, in words, for the inverse of this function, and then translate the verbal steps for the inverse into a formula for the inverse function.

10. Find a formula for the inverse function of \( f(x) = 3x - 2 \). Verify that \( f^{-1}(f(5)) = 5 \) and \( f(f^{-1}(2)) = 2 \).

11. Find a formula for the inverse function of \( g(x) = 2x + 1 \). Verify that \( g^{-1}(g(1)) = 1 \) and \( g(g^{-1}(7)) = 7 \).

12. Find a formula for the inverse function of \( h(x) = 2e^{3x} \). Verify that \( h^{-1}(h(0)) = 0 \).

13. Find a formula for the inverse function of \( w(x) = 5 + \ln(x) \). Verify that \( w^{-1}(w(1)) = 1 \).

14. If the graph of \( f \) goes through the point \((2, 5)\) and \( f'(2) = 3 \), then the graph of \( f^{-1} \) goes through the point \((-\_, \_\_)\) and \((f^{-1})'(5) = \_\_\_\_\_\_\) ?

15. If the graph of \( f \) goes through the point \((1, 3)\) and \( f'(1) > 0 \), then the graph of \( f^{-1} \) goes through the point \((-\_, \_\_)\). What can be said about \((f^{-1})'(3)\) ?

16. If \( f(6) = 2 \) and \( f'(6) < 0 \), then the graph of \( f^{-1} \) goes through the point \((-\_, \_\_)\). What can be said about \((f^{-1})'(6)\) ?

17. If \( f'(x) > 0 \) for all values of \( x \), what can be said about \((f^{-1})'(x)\)? What does this mean about the graphs of \( f \) and \( f^{-1} \)?

18. If \( f'(x) < 0 \) for all values of \( x \), what can be said about \((f^{-1})'(x)\)? What does this mean about the graphs of \( f \) and \( f^{-1} \)?

19. Find a linear function \( f(x) = ax + b \) so the graphs of \( f \) and \( f^{-1} \) are parallel and do not intersect.

20. Does \( f(x) = 3 + \sin(x) \) have an inverse function? Justify your answer.

21. Does \( f(x) = 3x + \sin(x) \) have an inverse function? Justify your answer.

22. For which positive integers \( n \) is \( f(x) = x^n \) one-to-one? Justify your answer.

23. Some functions are their own inverses. For which four of these functions does \( f^{-1}(x) = f(x) \)?
(a) \( f(x) = \frac{1}{x} \)  
(b) \( f(x) = \frac{x + 1}{x - 1} \)  
(c) \( f(x) = \frac{3x - 5}{7x - 3} \)  
(d) \( f(x) = x + a \ (a \neq 0) \)  
(e) \( f(x) = \frac{ax + b}{cx - a} \)
Reflections on Folding  (Optional)

The symmetry of the graphs of a function and its inverse about the line $y = x$ make it easy to sketch the graph of the inverse function from the graph of the function: we just fold the graph paper along the line $y = x$, and the graph of $f^{-1}$ is the "folded" image of $f$. The simple idea of obtaining a new image of something by folding along a line can enable us to quickly "see" solutions to some otherwise difficult problems.

The minimum problem in Fig. 12 of finding the shortest path from town A to town B with an intermediate stop at the river is moderately difficult to solve using derivatives. Geometrically, it is quite easy (Fig. 13):

- obtain the point $B^*$ by folding the image of $B$ across the river line
- connect $A$ and $B^*$ with a straight line (the shortest path connecting $A$ and $B^*$)
- fold the $C$ to $B^*$ line back across the river to obtain the $A$ to $C$ to $B$ solution.

As an almost free bonus, we see that, for the minimum path, the angle of incidence at the river equals the angle of reflection from the river.

24. Devise an algorithm using "folding" to find the where at the bottom edge of the billiards table you should aim ball $A$ in Fig. 14 so that ball $A$ will hit ball $B$ after bouncing off the bottom edge of the table.

( Assume that the angle of incidence equals the angle of reflection.)

25. Devise an algorithm using "folding" to sketch the shortest path in Fig. 15 from town $A$ to town $B$ which includes a stop at the river and at the road. (One fold may not be enough.)
26. Devise an algorithm using "folding" to find where you should aim ball A at the bottom edge of the billiards table in Fig. 16 so that ball A will hit ball B after bouncing off the bottom edge and the right edge of the table. (Assume that the angle of incidence equals the angle of reflection. Unfortunately, in a real game of billiards, the ball picks up a spin, "English", when it bounces off the first bank, and then on the second bounce the angle of incidence does not equal the angle of reflection.)

The "folding" idea can even be useful if the path is not a straight line.

27. Fig. 17 shows the parabolic path of a thrown ball. If the ball bounces off the tall vertical wall in the Fig. 18, where will it land (hit the ground)? (Assume that the angle of incidence equals the angle of reflection and that the ball does not lose energy during the bounce.)

Sometimes "unfolding" a problem is useful too.

28. A spider and a fly are located at opposite corners of the cube as shown in Fig. 19. Sketch the shortest path the spider can take along the surface of the cube to reach the fly.
Section 7.1

PRACTICE Answers

<table>
<thead>
<tr>
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<th>g(x)</th>
<th>x</th>
<th>g⁻¹(x)</th>
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<td>0</td>
<td>4</td>
</tr>
<tr>
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</tr>
<tr>
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<td>4</td>
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</tr>
</tbody>
</table>

Practice 1: See Fig. 1.

Fig. 1: Tables for g and g⁻¹

Practice 2: See Fig. 2. If (a,a) is on the graph of g, then (a,a) is also on the graph of g⁻¹.

Fig. 2: Graphs of g and g⁻¹

Practice 3:
(a) \( y = f(x) = 2x - 5 \).
   
   Find \( f^{-1} \): \( x = 2y - 5 \) so \( y = \frac{x + 5}{2} \). \( f^{-1}(x) = \frac{x + 5}{2} \).

(b) \( y = g(x) = \frac{2x - 1}{x + 7} \).

   Find \( g^{-1} \): \( x = \frac{2y - 1}{y + 7} \) so \( xy + 7 = 2y - 1 \). Then \( y(x - 2) = -7x - 1 \) so \( y = \frac{-7x - 1}{x - 2} \).

   \( g^{-1}(x) = \frac{7x + 1}{2 - x} \).

(c) \( y = h(x) = 2 + \ln(3x) \).

   Find \( h^{-1} \): \( x = 2 + \ln(3y) \) so \( x - 2 = \ln(3y) \) and \( e^{x-2} = e^{\ln(3y)} \).

   Then \( 3y = e^{x-2} \) and \( y = \frac{1}{3} e^{x-2} \). \( h^{-1}(x) = \frac{1}{3} e^{x-2} \).

Practice 4:

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>f'(x)</th>
<th>f⁻¹(x)</th>
<th>(f⁻¹)’(x)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2</td>
<td>3</td>
<td>3</td>
<td>( \frac{1}{f'(3)} = \frac{1}{2} )</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>-2</td>
<td>2</td>
<td>( \frac{1}{f'(2)} = -1 )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>( \frac{1}{f'(0)} = \frac{1}{3} )</td>
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<td>2</td>
<td>1</td>
<td>( \frac{1}{f'(1)} = \frac{1}{2} )</td>
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</table>

Fig. 3: Completed Table 4