Section P.5 Solving Inequalities Algebraically and Graphically

Objective: In this lesson you learned how to solve linear inequalities, inequalities involving absolute values, polynomial inequalities, and rational inequalities.

I. Properties of Inequalities (Pages 54–55)

Solving an inequality in the variable $x$ means . . . finding all the values of $x$ for which the inequality is true.

Numbers that are solutions of an inequality are said to _____ satisfy _____ the inequality.

To solve a linear inequality in one variable, use the ____properties of inequalities _____ to isolate the variable.

When both sides of an inequality are multiplied or divided by a negative number, . . . the direction of the inequality symbol must be reversed.

Two inequalities that have the same solution set are _____ equivalent inequalities _______.

Complete the list of Properties of Inequalities given below.

1) Transitive Property: $a < b$ and $b < c \rightarrow a < c$

2) Addition of Inequalities: $a < b$ and $c < d \rightarrow a + c < b + d$

Important Vocabulary

Solutions of an inequality All values of the variable for which the inequality is true.

Graph of an inequality The set of all points on the real number line that represent the solution set of an inequality.

Linear inequality An inequality in one variable (usually $x$) that can be written in the form $ax + b < 0$ or $ax + b > 0$, where $a$ and $b$ are real numbers with $a \neq 0$.

Double inequality An inequality that represents two inequalities.

Critical numbers The $x$-values that make the polynomial in a polynomial inequality equal to zero.

Test intervals Open intervals along the real number line in which the polynomial has no sign changes.

What you should learn

How to recognize properties of inequalities
3) Addition of a Constant $c$: $a < b \rightarrow a + c < b + c$

4) Multiplication by a Constant $c$:
   
   For $c > 0$, $a < b \rightarrow ac < bc$
   
   For $c < 0$, $a < b \rightarrow ac > bc$

II. Solving a Linear Inequality (Pages 55–56)

Describe the steps that would be necessary to solve the linear inequality $7x - 2 < 9x + 8$.

Add 2 to each side. Subtract $9x$ from each side, and combine like terms. Divide each side by $-2$ and reverse the inequality. Write the solution set as an interval.

To use a graphing utility to solve the linear inequality $7x - 2 < 9x + 8$, graph $y_1 = 7x - 2$ and $y_2 = 9x + 8$ in the same viewing window. Use the intersect feature of the graphing utility to find the point of intersection. Noticing where the graph of $y_1$ lies below the graph of $y_2$, write the solution set.

The two inequalities $-10 < 3x$ and $14 \geq 3x$ can be rewritten as the double inequality $-10 < 3x \leq 14$.

III. Inequalities Involving Absolute Value (Page 57)

Let $x$ be a variable or an algebraic expression and let $a$ be a real number such that $a \geq 0$. The solutions of $|x| < a$ are all values of $x$ that ______ lie between $-a$ and $a$ ______. The solutions of $|x| > a$ are all values of $x$ that ______ are less than $-a$ or greater than $a$ ______.

Example 1: Solve the inequality: $|x + 11| - 4 \leq 0$

$[-15, -7]$

The symbol $\cup$ is called a ______ union ______ symbol and is used to denote ______ the combining of two sets ______.
Example 2: Write the following solution set using interval notation: \( x > 8 \) or \( x < 2 \)
\((-\infty, 2) \cup (8, \infty)\)

IV. Polynomial Inequalities (Pages 58–60)

Where can polynomials change signs?
Only at its zeros, the \( x \)-values that make the polynomial equal to zero.

Between two consecutive zeros, a polynomial must be . . .
entirely positive or entirely negative.

When the real zeros of a polynomial are put in order, they divide
the real number line into . . . intervals in which the
polynomial has no sign changes.

These zeros are the critical numbers of the inequality,
and the resulting open intervals are the test intervals.

Complete the following steps for determining the intervals on
which the values of a polynomial are entirely negative or entirely positive:

1) Find all real zeros of the polynomial, and arrange the
zeros in increasing order. The zeros of a polynomial are
its critical numbers.

2) Use the critical numbers of the polynomial to determine
its test intervals.

3) Choose one representative \( x \)-value in each test interval
and evaluate the polynomial at that value. If the value of
the polynomial is negative, the polynomial will have
negative values for every \( x \)-value in the interval. If the
value of the polynomial is positive, the polynomial will
have positive values for every \( x \)-value in the interval.

To approximate the solution of the polynomial inequality
\( 3x^2 + 2x - 5 < 0 \) from a graph, . . .
graph the associated polynomial \( y = 3x^2 + 2x - 5 \) and locate the portion of the graph
that is below the \( x \)-axis.
If a polynomial inequality is not given in general form, you should begin the solution process by writing the inequality in general form—with the polynomial on one side and zero on the other side.

**Example 3:** Solve $x^2 + x - 20 \geq 0$.

$(-\infty, -5] \cup [4, \infty)$

**Example 4:** Use a graph to solve the polynomial inequality $-x^2 - 6x - 9 > 0$.

$\emptyset$

V. Rational Inequalities (Page 61)

To extend the concepts of critical numbers and test intervals to rational inequalities, use the fact that the value of a rational expression can change sign only at its zeros and its undefined values. These two types of numbers make up the critical numbers of a rational inequality.

To solve a rational inequality, first write the rational inequality in standard form. Then find the zeros and undefined values of the resulting rational expression. Form the appropriate test intervals and test a point from each interval in the inequality. Select the test intervals that satisfy the inequality as the solution set.

**Example 5:** Solve $\frac{3x + 15}{x - 2} \leq 0$.

$[-5, 2)$