Section 2.6  Rational Functions and Asymptotes

Solutions to Even-Numbered Exercises

2. \( f(x) = \frac{5x}{x - 1} \)

(a) \[
\begin{array}{c|c|c|c|c|c}
 x & f(x) & x & f(x) & x & f(x) \\
 0.5 & -5 & 1.5 & 15 & 5 & 6.25 \\
 0.9 & -45 & 1.1 & 55 & 10 & 5.55 \\
 0.99 & -495 & 1.01 & 505 & 100 & 5.05 \\
 0.999 & -4995 & 1.001 & 5005 & 1000 & 5.005 \\
\end{array}
\]

(b) The zero of the denominator is \( x = 1 \), so \( x = 1 \) is a vertical asymptote. The degree of the numerator is equal to the degree of the denominator, so the line \( y = \frac{5}{1} = 5 \) is a horizontal asymptote.

(c) The domain is all real numbers except \( x = 1 \).

4. \( f(x) = \frac{3}{|x - 1|} \)

(a) \[
\begin{array}{c|c|c|c|c|c}
 x & f(x) & x & f(x) & x & f(x) \\
 0.5 & 6 & 1.5 & 6 & 5 & 0.75 \\
 0.9 & 30 & 1.1 & 30 & 10 & 0.33 \\
 0.99 & 300 & 1.01 & 300 & 100 & 0.03 \\
 0.999 & 3000 & 1.001 & 3000 & 1000 & 0.003 \\
\end{array}
\]

(b) The zero of the denominator is \( x = 1 \), so \( x = 1 \) is a vertical asymptote. Because the degree of the numerator is less than the degree of the denominator, the \( x \)-axis or \( y = 0 \) is a horizontal asymptote.

(c) The domain is all real numbers except \( x = 1 \).

6. \( f(x) = \frac{4x}{x^2 - 1} \)

(a) \[
\begin{array}{c|c|c|c|c|c|c}
 x & f(x) & x & f(x) & x & f(x) & x & f(x) \\
 0.5 & -2.66 & 1.5 & 4.8 & 5 & 0.833 & -5 & -0.833 \\
 0.9 & -18.95 & 1.1 & 20.95 & 10 & 0.40 & -10 & 0.40 \\
 0.99 & -199 & 1.01 & 201 & 100 & 0.04 & -100 & 0.04 \\
 0.999 & -1999 & 1.001 & 2001 & 1000 & 0.004 & -1000 & 0.004 \\
\end{array}
\]

(b) The zeros of the denominator are \( x = \pm 1 \) so both \( x = 1 \) and \( x = -1 \) are vertical asymptotes. Because the degree of the numerator is less than the degree of the denominator, the \( x \)-axis or \( y = 0 \) is a horizontal asymptote.

(c) The domain is all real numbers except \( x = \pm 1 \).
8. \( f(x) = \frac{1}{x - 3} \)
   - Vertical asymptote: \( x = 3 \)
   - Horizontal asymptote: \( y = 0 \)
   - Matches graph (d).

10. \( f(x) = \frac{1 - x}{x} \)
   - Vertical asymptote: \( x = 0 \)
   - Horizontal asymptote: \( y = -1 \)
   - Matches graph (e).

12. \( f(x) = \frac{-x + 2}{x + 4} \)
   - Vertical asymptote: \( x = -4 \)
   - Horizontal asymptote: \( y = -1 \)
   - Matches graph (f).

14. \( f(x) = \frac{3}{(x - 2)^3} \)
   - (a) Domain: all real numbers except \( x = 2 \)
   - (b) Vertical asymptote: \( x = 2 \)
   - Horizontal asymptote: \( y = 0 \)
   - [Degree of \( p(x) \) < degree of \( q(x) \)]

16. \( f(x) = \frac{2 - 5x}{2 + 2x} \)
   - (a) Domain: all real numbers except \( x = -1 \)
   - (b) Vertical asymptote: \( x = -1 \)
   - Horizontal asymptote: \( y = -\frac{5}{2} \)
   - [Degree \( p(x) = \) degree \( q(x) \)]

18. \( f(x) = \frac{3x^2 + 1}{x^2 + x + 1} \)
   - (a) Domain: All real numbers. The denominator has no real zeros. [Try the Quadratic Formula on the denominator.]
   - (b) Vertical asymptote: none
   - Horizontal asymptote: \( y = 3 \)
   - [Degree \( p(x) = \) degree \( q(x) \)]

20. \( f(x) = \frac{x^2(x - 3)}{x^2 - 3x} \), \( g(x) = x \)
   - (a) Domain of \( f \): all real numbers except 0 and 3
   - Domain of \( g \): all real numbers
   - (b) Because \( x^2 - 3x \) is a common factor of both the numerator and the denominator of \( f(x) \), neither \( x = 0 \) nor \( x = 3 \) is a vertical asymptote of \( f \). Thus, \( f \) has no vertical asymptotes.

   (c) 
<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-1</td>
<td>Undef.</td>
<td>1</td>
<td>2</td>
<td>Undef.</td>
<td>3.5</td>
<td>4</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3.5</td>
<td>4</td>
</tr>
</tbody>
</table>

   (d) \( f \) and \( g \) differ only where \( f \) is undefined.
22. \( f(x) = \frac{2x - 8}{x^2 - 9x + 20}, \quad g(x) = \frac{2}{x - 5} \)

(a) Domain of \( f \): all real numbers except 4 and 5
Domain of \( g \): all real numbers except 5

(b) Because \( x - 4 \) is a common factor of both the numerator and the denominator of \( f, x = 4 \) is not a vertical asymptote of \( f \). The only vertical asymptote is \( x = 5 \).

(c) \[
\begin{array}{c|ccccccc}
 x & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
-\frac{2}{5} & -\frac{1}{2} & -\frac{2}{5} & -1 & \text{Undef.} & \text{Undef.} & 2 \\
st & -\frac{2}{5} & -\frac{1}{2} & -\frac{2}{5} & -1 & -2 & \text{Undef.} & 2 \\
\end{array}
\]

(d) \( f \) and \( g \) differ only at \( x = 4 \) where \( f \) is undefined and \( g \) is defined.

24. \( f(x) = 2 + \frac{1}{x - 3} \)

(a) As \( x \to \pm\infty, f(x) \to 2 \).

(b) As \( x \to \infty, f(x) \to 2 \) but is greater than 2.

(c) As \( x \to -\infty, f(x) \to 2 \) but is less than 2.

26. \( f(x) = \frac{2x - 1}{x^2 + 1} \)

(a) As \( x \to \pm\infty, f(x) \to 0 \).

(b) As \( x \to \infty, f(x) \to 0 \) but is greater than 0.

(c) As \( x \to -\infty, f(x) \to 0 \) but is less than 0.

30. \( h(x) = 6 + \frac{4}{x^2 + 2} \)

There are no real zeros.

32. (a) \( C = \frac{25,000(15)}{100 - 15} \approx 4411.76 \)

The cost would be $4411.76.

(c) \( C = \frac{25,000(90)}{100 - 90} = 225,000 \)

The cost would be $225,000.

(e) No. The model is undefined for \( p = 100 \).

34. (a) Use data \((10, \frac{1}{7}), (20, \frac{1}{10}), (30, \frac{1}{11}), (40, \frac{1}{12}), (50, \frac{1}{13})\). The least squares line for this data \((x, 1/y)\) is:

\[
\frac{1}{y} = 0.164 - 0.0029x \implies y = \frac{1}{0.164 - 0.0029x} = \frac{1}{\frac{25260 - 447x}{154,000}} = \frac{154,000}{3(8420 - 149x)}
\]

(b) \[
\begin{array}{c|cccccc}
 x & 10 & 20 & 30 & 40 & 50 \\
 7.4 & 9.4 & 13.0 & 20.9 & 52.9 \\
\end{array}
\]

(c) No, the function is negative for \( x = 60 \).